A METHOD OF DEFINING STEADY-STATE RADIATIVE – CONDUCTIVE HEAT TRANSFER IN A PLANE-PARALLEL LAYER. I

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A method is given for solving for the steady-state radiative–conductive heat transfer in a planar layer of a medium; the radiation flux is represented by means of Green's functions.

A large amount of computer time is required in direct numerical solution of the equations for radiative–conductive transport in a nonscattering medium, so several methods have been proposed for transforming the initial equations to a form convenient for numerical realization. The methods may be divided into three groups in accordance with the ultimate form of the equations.

The first group involves conversion of the initial system to a nonlinear integral equation for the temperature distribution, the first instance of this appearing to be [1]. The treatment was extended [2] to boundaries with specular reflection. During the numerical solution, the integral term is replaced by a certain quadrature, after which the process involves only nonlinear algebraic equations for the temperature (in [3, 4], the system was linearized to give a system linear in $T_1 - T(x)$). The nonlinear system may be solved either by simple iteration [1, 5-9] or by Newton's method [10-14].

The second group [15-20] includes methods in which the radiation-transport equation obtained by integration (or else the equation for the derivative of the radiation flux [15, 16]) is substituted into the energy equation, which results in an integrodifferential equation of first or second order. This equation is solved numerically as follows. The temperature distribution (as initially specified or found by computation) is integrated numerically to define the integral radiation flux, which converts the energy equation to differential form. This is then solved [16, 18] by the Runge–Kutta method. Other finite-difference schemes have also been used [15, 17, 19]. The resulting temperature distribution is reused in the iteration scheme until the required accuracy is obtained. In [20], the linearization of the integral term was performed in the expression for the derivative of the radiation flux; the subsequent procedure involved finite-difference methods applied to the energy equation, with solution of the resulting system of linear equations.

The third group [21-25] consists of methods in which the initial transport and energy equations are employed directly. In [21, 22], a solution for the gray case was obtained by replacing these equations by a system of $2l_\mu + 2$ nonlinear differential first-order equations, which were then solved by quasilinearization. Another procedure was used in [23-25]. Initially, the temperature distribution was supplied, and the Runge–Kutta method was used to solve the differential equations, with the transport equation solved by finite-difference methods in terms of the wavelength and angle. The result for the radiation flux was substituted into the energy equation, which itself was integrated by the Runge–Kutta method. The process was repeated until the required accuracy was obtained.

These methods have been utilized in examining various phenomena in this area; although certain advances have been made, recent results show that very large amounts of machine time are required for detailed and full incorporation of factors such as selectivity in the medium and in the bounding surfaces, as well as the angle dependence of the reflection coefficient; this is so even if the results are of comparatively low accuracy. Here we give a new numerical method that allows one to incorporate these effects without excessive consumption of computer time.

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Consider a planar layer of partially transparent nonscattering material bounded by opaque specularly reflecting surfaces. The density of the total energy flux and the temperature of the hot boundary are known. We place the origin at the center of the layer and set the X axis along the flux direction. Then the problem amounts to solution of the transport equation

$$\cos\psi I'_\nu(x) = -\tau_\nu I_\nu(x) + \tau_\nu n_\nu^2 I_{e,\nu} \tag{1}$$

in conjunction with the energy equation

$$-(\Lambda/h)\,T'(x) + 2\pi \int_0^\infty \int_0^\pi I_\nu(x)\cos\psi\sin\psi\,d\nu\,d\psi = Q. \tag{2}$$

The boundary conditions take the form

$$I_\nu^+ = (1 - R_{1,\nu})\,n_\nu^2 I_{e,\nu} + R_{1,\nu} I_\nu^-, \quad x = -1, \tag{3}$$

$$I_\nu^- = (1 - R_{2,\nu})\,n_\nu^2 I_{e,\nu} + R_{2,\nu} I_\nu^+, \quad x = 1, \tag{4}$$

$$T(-1) = T_1. \tag{5}$$

We introduce the quantities

$$q_{\psi,\nu} = (I_\nu^+ - I_\nu^-)\,|\cos\psi|; \quad u_{\psi,\nu} = (I_\nu^+ + I_\nu^-)\,|\cos\psi|. \tag{6}$$

Note that $q_{\psi,\nu}$ is the density of the spectral direction flux, i.e., the total amount of energy transported per unit time in unit frequency range in unit solid angle through unit area perpendicular to the X axis by rays propagating in the directions (ψ, φ) and $(-\psi, \varphi)$.

The system of (1) and (2) takes the following form in terms of the new variables:

$$q'_{\psi,\nu}(x) = -\gamma_\nu u_{\psi,\nu}(x) + 2\tau_\nu n_\nu^2 I_{e,\nu}(x), \tag{7}$$

$$u'_{\psi,\nu}(x) = -\gamma_\nu q_{\psi,\nu}(x), \tag{8}$$

$$-(\Lambda/h)\,T'(x) + 2\pi \int_0^\infty \int_0^1 q_{\psi,\nu}(x)\,d\nu\,d\mu = Q. \tag{9}$$

We differentiate (7) and substitute the result into (8) to get

$$q''_{\psi,\nu}(x) - \gamma_\nu^2 q_{\psi,\nu}(x) = 2\tau_\nu n_\nu^2 I'_{e,\nu}(x). \tag{10}$$

The boundary conditions of (3) and (4) become

$$\begin{aligned} q'_{\psi,\nu}(x) - \gamma_\nu p_{1,\nu} q_{\psi,\nu}(x) = 0; \quad x = -1; \\ q'_{\psi,\nu}(x) + \gamma_\nu p_{2,\nu} q_{\psi,\nu}(x) = 0; \quad x = 1. \end{aligned} \tag{11}$$

We now have the system of (9) and (10) together with (5) and (11); this formulation is not equivalent to the initial one, since we do not have to calculate the spectral intensities I_ν^+ and I_ν^- in order to determine the temperature distribution, since we merely have to derive the less-detailed quantity $q_{\psi,\nu}$.

We express the solution to (10) subject to the boundary conditions of (11) in terms of the Green's function $g_\nu(x, \xi)$ for the corresponding homogeneous case:

$$q_{\psi,\nu}(x) = 2\tau_\nu n_\nu^2 \int_{-1}^{1} g_\nu(x, \xi)\, I'_{e,\nu}(\xi)\,d\xi, \tag{12}$$

$$g_\nu(x,\xi) = \beta_\nu [R_{1,\nu} e^{-\gamma_\nu(2+x+\xi)} + R_{2,\nu} e^{-\gamma_\nu(2-x-\xi)} - R_{1,\nu} R_{2,\nu} e^{-\gamma_\nu(4+x-\xi)} - e^{-\gamma_\nu|x-\xi|}]; \quad -1 \leqslant x < \xi,$$

$$g_\nu(x,\xi) = \beta_\nu [R_{1,\nu} e^{-\gamma_\nu(2+x+\xi)} + R_{2,\nu} e^{-\gamma_\nu(2-x-\xi)} - R_{1,\nu} R_{2,\nu} e^{-\gamma_\nu(4-x+\xi)} - e^{-\gamma_\nu|x-\xi|}]; \quad \xi < x \leqslant 1. \tag{13}$$

We introduce the dimensionless quantities

$$\theta = T/T_1, \quad N = \Lambda/(\bar{n}^2 \sigma T_1^3 h), \quad a = Q/(\bar{n}^2 \sigma T_1^4), \tag{14}$$

where θ, N, and a are the dimensionless temperature, dimensionless thermal conductivity, and dimensionless total energy flux density, which gives

$$-N\theta'(x) + \frac{2\pi}{n^2\sigma T_1^4}\int_0^\infty\int_0^1 q_{\psi,\nu}(x)\,d\nu\,d\mu = a, \tag{15}$$

which is used with (12) and the boundary condition $\theta(-1) = 1$ to define the temperature distribution and the spectral radiation fluxes in the system.

These steps have reduced the number of independent parameters by one (the half-thickness of the layer h has disappeared). If the medium has selective optical characteristics (selective case), then the numerical representation for a single value τ_ν results in l_ν independent parameters (numbers). Since l_ν is fairly large for a reasonably precise approximation, there is virtually no advantage from this reduction in the number of parameters.

The situation is different for a grey medium (gray problem). Equations (12) and (15) (the latter after integration of $q_{\psi,\nu}$ with respect to frequency) show that the number of parameters is then reduced substantially; for instance, for symmetrical boundaries we have instead of the seven parameters (n, k, R, Λ, h, T_1, Q), the four parameters (τ, R, N, a).

Complete analysis of the heat transport in a planar gray medium ceases to be almost hopeless when these dimensionless parameters are used; for instance, numerical realization of this method requires perhaps 1.5-2 h of BÉSM computer run time to calculate a series of temperature distributions and radiation fluxes for certain ranges in τ and R for all possible sets of N and a (about 10 points in N and about 10 points in a). The results from such calculations allow one to predict roughly what picture will be observed for a real nongray medium. This latter feature, rather than the desire to reduce the number of independent parameters, is the basis for using dimensionless quantities in selective problems.

There are various ways of introducing dimensionless parameters, and one of the basic requirements here is to provide the closest possible correspondence between the selective and gray cases. Of course, complete identity is not possible because in the selective case one cannot identify a parameter that corresponds adequately to the optical thickness 2τ in the gray case. The other quantities defined by (14) meet this requirement if we neglect the weak wavelength dependence of the refractive index. Most studies on such heat transport for gray media have $N^* = \Lambda k/(n^2\sigma T_1^3)$ as the analog of the parameter N, or some similar quantity. It will be clear from the above why such a quantity is not suitable for the selective case. If, on the other hand, we use a quantity related to the geometrical dimensions [15, 23, 26, 27] in producing the dimensionless parameters, then the meaning of N in the selective case will be the same as in the gray one.

Another requirement imposed on the dimensionless quantities is that the dimensionless energy Eq. (15) should have the same form as the dimensional one. Of course, use of the half-thickness h as the divisor in producing the dimensionless quantity causes the range in the independent variable x to be [-1, 1].

It is clear that the nonlinearity arises from T(x), so it would be extremely valuable in each iteration to use only functions of x, all operations with the variables ν and μ having been completed previously. This can be done by uncoupling the variables ν and ξ in the expression for the Planck function, which is expanded around some temperature T_0 as a Taylor series. Formal integration of (15) then gives us a system of two nonlinear integral equations whose kernels are dependent on the Green's function:

$$\theta(x) = 1 - (a/N)(1 + x) + \sum_i \int_{-1}^{1} J_i(x,\ \xi)\hat{\eta}_i\theta(\xi)\,d\xi; \quad i = 1,\ 2,\ \ldots,\ m, \tag{16}$$

$$\theta'(x) = -a/N + \sum_i \int_{-1}^{1} G_i(x,\ \xi)\,\hat{\eta}_i\theta(\xi)\,d\xi, \tag{17}$$

where

$$J_i(x,\ \xi) = \int_0^\infty\int_0^1 \tau_\nu n_\nu^2 I_{e,\nu}^{(i)}(T_0)\,j_\nu(x,\xi)\,d\nu\,d\mu, \tag{18}$$

$$G_i(x,\ \xi) = \int_0^\infty\int_0^1 \tau_\nu n_\nu^2 I_{e,\nu}^{(i)}(T_0)\,g_\nu(x,\ \xi)\,d\nu\,d\mu, \tag{19}$$

$$j_\nu(x,\ \xi) = \int_{-1}^{x} g_\nu(\xi',\ \xi)\, d\xi', \tag{20}$$

$$\hat{\eta}_i\theta = (4\pi/N\bar{n}^2\sigma T_1^3)\,[T_1(\theta - \theta_0)]^{i-1}\,\theta'/(i-1)!, \quad \theta_0 = T_0/T_1, \tag{21}$$

and $I^{(i)}_{e,\nu}$ is the derivative of order i with respect to temperature at $T = T_0$.

A very important feature is that in the case of symmetrical boundaries, $R_1 = R_2$, the Green's function has the following symmetry features: $g_\nu(x, \xi) = g_\nu(-x, -\xi)$, $g_\nu(x, \xi) = g_\nu(\xi, x)$; although the original Green's function is not symmetrical, it is readily shown that the following expansion applies:

$$j_\nu(x,\ \xi) = \tilde{j}_\nu(x,\ \xi) + \varphi_\nu(\xi), \text{ where } \tilde{j}_\nu(x,\ \xi) = -\tilde{j}_\nu(-x,\ -\xi).$$

The symmetry reduces the executive store required very considerably and also nearly halves the run time.

The above procedure for deriving the equations for the temperature distribution is not specific to this formulation: similar schemes can be given for other similar problems, in particular, nonstationary ones.

Aspects of the numerical realization and certain results will be given in the second part of the paper.

NOTATION

x or ξ	is the coordinate of a point in layer ($\in[-1, 1]$);
ψ	is the angle reckoned from the positive direction of the X axis;
$\mu = [\cos\psi]$; φ	is the azimuth;
I	is the radiation intensity;
I^+, I^-	are the radiation intensities in the directions for which $0 \le \psi < \pi/2$ and $\pi/2 \le \psi \le \pi$, respectively;
T	is the temperature;
$\theta = T/T_1$;	
Q	is the energy flux density;
a	is the dimensionless total energy flux density in (14);
$q_{\psi,\nu}$	is the spectral directional flux density;
n	is the refractive index;
n^2	is the refractive index averaged over the spectrum;
k	is the absorption coefficient;
h	is the half-thickness of layer;
$\tau = kh$	
R	is the reflection coefficient;
Λ	is the thermal conductivity;
N	is the dimensionless thermal conductivity in (14);
$\gamma = \tau/\mu$; $\beta = 1/[2\gamma(1 - R_1R_2e^{-4\gamma})]$; $p = (1 + R)/(1 - R)$;	
m	is the number of terms in the expansion of $I^1_{e,\nu}(T)$ as a Taylor series;
l	is the number of nodes in the quadratic formula.

Indices

e	is the equilibrium radiation in vacuum;
ν	is the frequency;
ψ or μ	is the angle;
1, 2	are the hot and cold boundaries, respectively;
0	is the temperature used in the Taylor series.

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A PROBLEM IN THERMAL-CONDUCTIVITY THEORY

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A solution of the problem of heat distribution in an infinite lamina is presented.

Let an infinite lamina of thickness $2l(-l \leq x \leq l)$ have an initial temperature u_0. Over the course of a time t_1 it is heated by a constant thermal flux of density q, as a result of which the temperature of the surfaces bounding the lamina becomes equal to u_1. It is required to determine by what law the thermal flux must change further in order that the lamina surfaces be maintained at this temperature u_1. Initially, we find the temperature distribution law at the end of heating, i.e., after expiration of time t_1. To do this we use the solution of the thermal-conductivity equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{1}$$

with initial condition

$$u(x, 0) = u_0 \tag{2}$$

and boundary conditions

$$-\lambda \frac{\partial u(x, t)}{\partial x}\Big|_{x=-l} = q, \tag{3}$$

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